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AXISYMMETRIC HARMONIC INTERPOLATION POLYNOMIALS IN R^N

BY

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ABSTRACT. Corresponding to a given function $F(x, \rho)$ which is axisymmetric harmonic in an axisymmetric region $\Omega \subset R^3$ and to a set of $n+1$ circles C_n in an axisymmetric subregion $A \subset \Omega$, an axisymmetric harmonic polynomial $\Lambda_n(x, \rho; C_n)$ is found which on the C_n interpolates to $F(x, \rho)$ or to its partial derivatives with respect to x. An axisymmetric subregion $B \subset \Omega$ is found such that $\Lambda_n(x, \rho; C_n)$ converges uniformly to $F(x, \rho)$ on the closure of B. Also a $\Lambda_n(x, \rho; x_0, \rho_0)$ is determined which, together with its first n partial derivatives with respect to x, coincides with $F(x, \rho)$ on a single circle (x_0, ρ_0) in Ω and converges uniformly to $F(x, \rho)$ in a closed torus with (x_0, ρ_0) as central circle.

1. Introduction. In this paper we study the interpolation of an axisymmetric harmonic function by means of axisymmetric harmonic polynomials. We choose the x-axis as our line of symmetry and use mainly cylindrical coordinates (x, ρ, ϕ) related to rectangular coordinates (x, y, z) by the equations $y = \rho \cos \phi$, $z = \rho \sin \phi$ and to spherical coordinates (r, θ, ϕ) by the equations $x = r \cos \theta$, $\rho = r \sin \theta$. By a circle (x_k, ρ_k) we mean the circle with the equations $x = x_k$, $\rho = \rho_k$. Furthermore, we define a region Ω to be axisymmetric if $(x_0, \rho_0, \phi_0) \in \Omega$ implies that also $(x_0, \rho, \phi) \in \Omega$ for all $0 \le \rho \le \rho_0$ and $0 \le \phi \le 2\pi$. We define a function F to be axisymmetric in Ω if its values $F(x, \rho)$ in Ω are independent of ϕ.

Specifically, we deal with the following two general problems for a given axisymmetric region Ω, a given function $F(x, \rho)$ axisymmetric harmonic in Ω and a given set of circles $C_n = \{(x_k, \rho_k);\ k = 0, 1, \cdots, n\}$.

(1) To find an axisymmetric harmonic polynomial $\Lambda_n(x, \rho; C_n)$ of degree n such that

$$\Lambda_n(x_k, \rho_k; C_n) = F(x_k, \rho_k), \qquad k = 0, 1, \cdots, n. \tag{1.1}$$

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(2) To find two axisymmetric subregions A and B of Ω, with closures $\bar{A} \subset \Omega$ and $\bar{B} \subset \Omega$, such that, if $C_n \subset \bar{A}$, then $\Lambda_n(x, \rho; C_n)$ converges to $F(x, \rho)$ uniformly in $\bar{B}$.

Such problems may be interpreted as those of determining empirically the velocity potential $F(x, \rho)$ in an axisymmetric flow of an incompressible fluid. An example is that of a liquid or gas which flows from $x = -\infty$ in a uniform stream parallel to the x-axis, but which eventually streams past a smooth axisymmetric obstacle K. Region Ω is chosen as disjoint from K and its interior. The values $F(x_k, \rho_k)$ might be found from measurements of the velocity potential taken along a set of circles $C_n = \{(x_k, \rho_k)\}$ within $\bar{A} \subset \Omega$, and the interpolating polynomial $\Lambda_n(x, \rho; C_n)$ would then be sought as satisfying equations (1.1). The subregions A and B of Ω must then be selected so that $\Lambda_n(x, \rho; C_n)$ would approximate to the potential $F(x, \rho)$ uniformly in $\bar{B}$ to within a prescribed degree of accuracy. (See also [4, pp. 254–256]; [6, pp. 432–467].)

Our methods for solving the first type of problem involve straightforward algebraic procedures. However, our method for solving the second type of problem is by use of the Whittaker-Bergman operator [1, pp. 43–57]

$$G(x, y, z) = \frac{1}{2\pi i} \int_{|\tau|=1} g(\zeta, \tau)\tau^{-1}\, d\tau, \tag{1.2}$$

where $\zeta = x + (i/2)y(\tau + \tau^{-1}) + (1/2)z(\tau - \tau^{-1})$, which generates a harmonic function $G(x, y, z)$ as the transform of a function $g(\zeta, \tau)$ that is holomorphic in ζ over some region in $\mathbf{C}$ and continuous in τ for $|\tau| = 1$. (In general, $G(x, y, z)$ is complex valued so that its real and imaginary parts are separately harmonic.) By this device, we are able to derive some results on harmonic polynomial interpolation in $\mathbf{R}^3$ analogous to certain results about polynomial interpolation in $\mathbf{C}$.

Finally, by similar methods, we are able to generalize our results to axisymmetric harmonic functions in $\mathbf{R}^N$.

2. Interpolating polynomial $\Lambda_n(x, \rho; C_n)$. As an axisymmetric harmonic polynomial, $\Lambda_n(x, \rho; C_n)$ may be written as a linear combination [4, p. 254] of the zonal harmonics $r^k P_k(x/r)$ where $P_k(u)$ is the kth degree Legendre polynomial. As is well known [4, p. 125]

$$r^k P_k(x/r) = \sum_{j=0}^{[k/2]} (-1)^j \gamma_{k-j} x^{k-2j} r^{2j}$$

where $[k/2]$ is the largest integer $j \leq k/2$ and

$$\gamma_{kj} = [1 \cdot 3 \cdots (2k - 2j - 1)]/[j!(k - 2j)!2^j].$$

Substituting $r^2 = x^2 + \rho^2$, we define

$$P_k(x, \rho) = r^k P_k(x/r). \tag{2.1}$$

Thus $P_k(x, \rho)$ is an axisymmetric harmonic polynomial which is a homogeneous function of order k in x and ρ. It is an even function of ρ but is an odd or even function of x according as k is an odd or even integer. Furthermore, since $P_k(1) = 1$, we find $P_k(1, 0) = 1$ for all k. In short we may write

$$\Lambda_n(x, \rho; C_n) = \sum_{k=0}^{n} A_k P_k(x, \rho). \tag{2.2}$$

The coefficients A_k are to be determined so that Equation (1.1) holds; that is, so as to satisfy the system

$$\sum_{k=0}^{n} A_k P_k(x_j, \rho_j) = F(x_j, \rho_j), \qquad j = 0, 1, \cdots, n. \tag{2.3}$$

Eliminating the A_k from (2.2) and system (2.3), we obtain for Λ_n the equation

$$\begin{vmatrix} \Lambda_n(x, \rho; C_n) & 1 & P_1(x, \rho) & \cdots & P_n(x, \rho) \\ F(x_0, \rho_0) & 1 & P_1(x_0, \rho_0) & \cdots & P_n(x_0, \rho_0) \\ F(x_1, \rho_1) & 1 & P_1(x_1, \rho_1) & \cdots & P_n(x_1, \rho_1) \\ \vdots & \vdots & \vdots & & \vdots \\ F(x_n, \rho_n) & 1 & P_1(x_n, \rho_n) & \cdots & P_n(x_n, \rho_n) \end{vmatrix} = 0. \tag{2.4}$$

Using the notation

$$V(C_n) = \det \|P_k(x_j, \rho_j)\|, \qquad j, k = 0, 1, \cdots, n; \tag{2.5}$$

$$V_k(x, \rho; C_n) = [V(C_n)]_{(x_k, \rho_k) = (x, \rho)} \tag{2.6}$$

for $k = 0, 1, \cdots, n$, we solve (2.4) explicitly for Λ_n to obtain

$$\Lambda_n(x, \rho; C_n) = \left[\sum_{k=0}^{n} F(x_k, \rho_k) V_k(x, \rho; C_n)\right] \Big/ V(C_n) \tag{2.7}$$

provided that $V(C_n) \neq 0$.

The restriction $V(C_n) \neq 0$ on the choice of the circles $C_n = \{(x_k, \rho_k)\}$ implies first that the circles (x_k, ρ_k) are distinct. It implies secondly that the equations

$$P_k(x_j, \rho_j) = 0 \tag{2.8}$$

are not satisfied for any k simultaneously for all $j = 0, 1, \cdots, n$. Factoring the Legendre polynomial

$$P_k(u) = \gamma_k \prod_{\nu=1}^{k} (u - \cos \alpha_{k\nu})$$

and thus

$$(2.9) \qquad P_k(x, \rho) = \gamma_k \prod_{\nu=1}^{k} (x - r \cos \alpha_{k\nu}),$$

we may interpret (2.8) as requiring that each circle (x_j, ρ_j) lie on some cone

$$(2.10) \qquad \rho = x \tan \alpha_{k\nu}$$

for the same value of k. Also, for given (x_j, ρ_j), $j = 1, 2, \cdots, n$, and given ρ_0, we may determine the zeros of $V(C_n)$ considered as an nth degree polynomial in x_0 and thereby find possibly additional sets of circles C_n which fail to satisfy the condition $V(C_n) \neq 0$.

3. **Special cases.** Let us first consider the case that all the circles degenerate into points on the axis of symmetry so that $\rho_0 = \rho_1 = \cdots = \rho_n = 0$. Since now $r_k^2 = x_k^2$, we have $P_j(x_k, 0) = x_k^j P_j(1) = x_k^j$ and thus

$$V(C_n) = v(x_0, x_1, \cdots, x_n),$$

$$V_k(x, \rho; C_n) = \sum_{j=0}^{n} v_{jk}(x_0, \cdots, x_n) P_j(x, \rho),$$

where

$$(3.1) \qquad v(x_0, x_1, \cdots, x_n) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix},$$

the Vandermonde determinant and $v_{jk}(x_0, x_1, \cdots, x_n)$ is the cofactor of x_k^j in $v(x_0, x_1, \cdots, x_n)$. Thus

$$(3.2) \qquad \Lambda_n(x, \rho; C_n) = \sum_{j,k=0}^{n} \left[\frac{v_{jk}(x_0, \cdots, x_n)}{v(x_0, \cdots, x_n)} \right] F(x_k, 0) P_j(x, \rho).$$

On the x axis $\Lambda_n(x, 0; C_n)$ is found from (3.2) to reduce to the Lagrange interpolation polynomial for $\mathcal{F}(x) = F(x, 0)$.

Let us next examine the special case $x_0/r_0 = x_1/r_1 = \cdots = x_n/r_n = u_0 = \cos \alpha$ when the circles C_n lie on the cone $r = (x/u_0)$, but not on any cone (2.10);

that is, $\alpha \neq \alpha_{k\nu}$, $\nu = 1, 2, \cdots, k$; $k = 1, \cdots, n$. In this case

$$P_k(x_j, \rho_j) = r_j^k P_k(u_0),$$
$$V(C_n) = P_1(u_0)P_2(u_0) \cdots P_n(u_0)v(r_0, r_1, \cdots, r_n),$$
$$V_k(x, \rho; C_n) = \sum_{j=0}^{n} P_j(x, \rho)P_1(u_0) \cdots P_{j-1}(u_0) \cdot P_{j+1}(u_0) \cdots P_n(u_0)v_{jk}(r_0, \cdots, r_n).$$

Thus,

$$(3.3)\quad \Lambda_n(x, \rho; C_n) = \sum_{j,k=0}^{n} \left[\frac{v_{jk}(r_0, \cdots, r_n)}{v(r_0, \cdots, r_n)}\right]\left[\frac{P_j(x, \rho)}{P_j(u_0)}\right] F(x_k, x_k \tan \alpha).$$

Along a circle on the cone, $\Lambda_n(x, x \tan \alpha; C_n)$ is found from (3.3) to reduce to the Lagrange interpolation polynomial to $\mathcal{F}(x) = F(x, x \tan \alpha)$.

Finally, let us look into the special case $r_0 = r_1 = \cdots = r_n$ when all the circles C_n lie on the sphere $x^2 + \rho^2 = r_0^2$. Since now $P_k(x, \rho) = r_0^k P_k(u)$,

$$(3.4)\qquad V(C_n) = r_0^{n(n+1)/2} \det \|P_k(u_j)\|.$$

Let us write $P_n(u) = \gamma_n u^n + \sum_{\nu=0}^{n-1} \mu_{n\nu} P_\nu(u)$. By subtracting from the kth column of the determinant in (3.4) for each k, suitable linear combinations of the first $(k-1)$ columns, we may reduce (3.4) to the form

$$(3.5)\qquad V(C_n) = r_0^{n(n+1)/2} \gamma_1 \gamma_2 \cdots \gamma_n v(u_0, u_1, \cdots, u_n).$$

Similarly

$$V_k(x, \rho; C_n) = \gamma_1 \gamma_2 \cdots \gamma_n \sum_{i=0}^{n} \gamma_j^{-1} r_0^{[n(n+1)-2j]/2} r_{jk}$$

where

$$r_{jk} = \{v_{jk}(u_0, \cdots, u_n) - \mu_{j+1}\gamma_{j+1}^{-1} v_{j+1,k}(u_0, \cdots, u_n)\}.$$

Thus

$$\Lambda_n(x, \rho; C_n) = \sum_{j,k=0}^{n} [r_0^j \gamma_j v(u_0, \cdots, u_n)]^{-1} r_{jk} F(x_k, \rho_k) P_j(x, \rho).$$

Along a circle on the sphere $r = r_0$, $\Lambda_n(x, \rho; C_n)$ reduces to the Lagrange interpolation polynomial for $\mathcal{F}(x) = F(x, [r_0^2 - x^2]^{1/2})$.

4. Integral representations for $\Lambda_n(x, \rho; C_n)$. In view of the Laplace formula for Legendre polynomials,

$$P_k(x, \rho) = \frac{1}{\pi}\int_0^\pi (x + i\rho \cos t)^k \, dt, \tag{4.1}$$

we may rewrite (2.2) in the form

$$\Lambda_n(x, \rho; C_n) = \frac{1}{\pi}\int_0^\pi \lambda_n(x + i\rho \cos t)\, dt \tag{4.2}$$

where the polynomial

$$\lambda_n(\zeta) = \sum_{k=0}^{n} A_k \zeta^k, \qquad \zeta \in \mathbf{C}, \tag{4.3}$$

is called the associate of $\Lambda_n(x, \rho; C_n)$. More generally, if $F(x, \rho)$ is an axisymmetric harmonic function in an axisymmetric region $\Omega \subset \mathbf{R}^3$, then

$$F(x, \rho) = \frac{1}{\pi}\int_0^\pi f(x + i\rho \cos t)\, dt, \tag{4.4}$$

where the associate $f(\zeta) = F(\zeta, 0)$ is a holomorphic function of ζ in the meridian cross-section ω of Ω, obtained on intersecting Ω with any plane through the x axis. This meridian section ω is an axiconvex region in $\mathbf{C}$, in the sense that $\zeta \in \omega$ implies that also $[\mu\zeta + (1-\mu)\overline{\zeta}] \in \omega$ for all μ, $0 \le \mu \le 1$. In fact, (4.4) is the special case of equation (1.2) with $g(\zeta, \tau) = f(\zeta)$.

Using (4.1), (4.2) and (4.3), we may now rewrite (2.5) in the form

$$V(C_n) = \pi^{-n-1}\int_0^\pi \int_0^\pi \cdots \int_0^\pi v(\sigma_0, \sigma_1, \cdots, \sigma_n)\, dt_0\, dt_1 \cdots dt_n$$

where $\sigma = x + i\rho \cos t$, $\sigma_k = x_k + i\rho_k \cos t_k$, $k = 0, 1, \cdots, n$. Let us introduce the closed $n+1$ and $n+2$ dimensional cubes

$$T = \{t_0, t_1, \cdots, t_n\}, \qquad 0 \le t_k \le \pi, \quad k = 0, 1, \cdots, n,$$
$$T^* = \{t, t_0, t_1, \cdots, t_n\}, \qquad 0 \le t \le \pi, \quad 0 \le t_k \le \pi, \ k = 0, 1, \cdots, n$$

and the notation

$$dT = dt_n \cdots dt_1\, dt_0, \qquad dT^* = dt_n \cdots dt_1\, dt_0\, dt,$$
$$S = \{\sigma_0, \sigma_1, \cdots, \sigma_n\}, \qquad v(S) = v(\sigma_0, \sigma_1, \cdots, \sigma_n),$$
$$T_k = (T)_{t_k = t}, \qquad S_k = (S)_{\sigma_k = \sigma}.$$

Thus we may write

(4.5) $$V(C_n) = \pi^{-n-1} \int_T v(S)\,dT$$

and similarly

$$F(x_k, \rho_k)V_k(x, \rho; C_n) = \pi^{-n-2} \int_{T^*} f(\sigma_k)v(S_k)\,dT^*.$$

Consequently, from (2.7) it follows that

(4.6) $$\Lambda_n(x, \rho; C_n) = \frac{\sum_{k=0}^{n} \int_{T^*} f(\sigma_k)v(S_k)\,dT^*}{\pi \int_T v(S)\,dT}.$$

5. Relation of Λ_n to the Lagrange interpolation polynomial for $f(\sigma)$. Let $l_n(\sigma; S)$ be the Lagrange polynomial which interpolates to $f(\sigma)$ at the points $\sigma_k = x_k + i\rho_k \cos t_k$, $k = 0, 1, \cdots, n$. As is well known,

$$l_n(\sigma; S) = \sum_{k=0}^{n} \frac{f(\sigma_k)\psi(\sigma)}{\psi'(\sigma_k)(\sigma - \sigma_k)}$$

where $\psi(\sigma) = (\sigma - \sigma_0)(\sigma - \sigma_1) \cdots (\sigma - \sigma_n)$.

Equivalently,

(5.1) $$l_n(\sigma; S) = \sum_{k=0}^{n} \frac{f(\sigma_k)v(S_k)}{v(S)},$$

whereupon (4.6) becomes

(5.2) $$\Lambda_n(x, \rho; C_n) = \frac{\int_{T^*} l_n(\sigma; S)v(S)\,dT^*}{\pi \int_T v(S)\,dT}.$$

Thus we may regard $\Lambda_n(x, \rho; C_n)$ as a certain average of the values of $l_n(\sigma; S)$ taken over the cube T^*.

6. Approximation of $\Lambda_n(x, \rho\,; C_n)$ to $F(x, \rho)$. Let us derive an integral representation for the difference

(6.1) $$\Delta_n(x, \rho; C_n) = F(x, \rho) - \Lambda_n(x, \rho; C_n).$$

Rewriting (4.4) as $F(x, \rho) = \int_{T^*} f(\sigma)v(S)\,dT^*/\pi \int_T v(S)\,dT$, we infer from (5.2) that

(6.2) $$\Delta_n(x, \rho; C_n) = \frac{\int_{T^*} [f(\sigma) - l_n(\sigma, S)]v(S)\,dT^*}{\pi \int_T v(S)\,dT}.$$

We now state the following:

Theorem I. *Let Ω be an axisymmetric region and A and B axisymmetric subregions whose closures $\bar{A}$ and $\bar{B}$ lie in Ω. Let ω, α and β be respectively* *the meridian sections of* Ω, A *and* B. *Let* $F(x, \rho)$ *be an axisymmetric harmonic function in* Ω *and let* $C_n = \{(x_k, \rho_k)\}$ *be a set of* $n+1$ *circles in* $\overline{A}$ *with* $x_0 < x_1 < \cdots < x_n$. *Let*

$$\mathfrak{X}(C_n) = \int_T |\nu(S)| dT \Big/ \left| \int_T \nu(S)\, dT \right|. \tag{6.3}$$

Assume further that A *and* B *have the properties*:

(1) *there exists a constant* $M \geq 1$ *and an integer* $N > 0$ *such that* $\mathfrak{X}(C_n) \leq M$ *for all sets* $C_n \subset \overline{A}$, $n \geq N$;

(2) $l_n(\sigma, S)$ *approximates* $f(\sigma)$ *uniformly for all* $\sigma \in \overline{\beta}$ *and all* T *with* $0 \leq t_k \leq \pi$, $k = 0, 1, \cdots, n$. *Then* $\Lambda_n(x, \rho; C_n)$ *approximates uniformly to* $F(x, \rho)$ *for all* $(x, \rho) \in \overline{B}$.

Proof. From (6.2) we infer that

$$|\Delta_n(x, \rho; C_n)| \leq \frac{\int_{T^*} |f(\sigma) - l_n(\sigma, S)|\,|\nu(S)|\, dT^*}{\pi \left| \int_T \nu(S)\, dT \right|}. \tag{6.4}$$

By hypothesis (2), given an $\epsilon > 0$, we can find $N_0 > 0$ so that $|f(\sigma) - l_n(\sigma, S)| < \epsilon/M$ for all $n \geq N_0$, for all $\sigma \in \overline{\beta}$ and all T with $0 \leq t_k \leq \pi$, $k = 0, 1, \cdots, n$. Using also hypothesis (1), we find

$$|\Delta_n(x, \rho; C_n)| < \frac{\epsilon \int_{T^*} |\nu(S)| dT^*}{M\pi \left| \int_T \nu(S)\, dT \right|} < \epsilon$$

for all $(x, \rho) \in \overline{B}$ and all $n \geq \max(N_0, N)$, thereby completing the proof of Theorem I.

7. Sufficient conditions on subregions A **and** B. Let us first examine some sufficient conditions on A for the existence of a constant $M \geq 1$ such that $\mathfrak{X}(C_n) \leq M$. It is clear from (6.3) that in any case $\mathfrak{X}(C_n) \geq 1$. If $\rho_0 = \rho_1 = \cdots = \rho_n = 0$, then

$$\nu(S) = \prod_{k=1}^{n} \prod_{j=0}^{k-1} (x_k - x_j) = |\nu(S)| > 0$$

and $\mathfrak{X}(C_n) = 1$. Thus, if we are given any constant $M \geq 1$, we can make $\mathfrak{X}(C_n) \leq M$ by taking, if necessary, all the ρ_k sufficiently small.

Alternatively, let us choose C_n so that for some δ, $0 < \delta < \pi/2$,

$$|\arg \nu(S)| \leq \delta. \tag{7.1}$$

Then

$$\left|\int_T v(S)\,dT\right| \geq \Re\left[\int_T v(S)\,dT\right] \geq \int_T \Re v(S)\,dT$$
$$\geq \int_T |v(S)| \cos \arg v(S)\,dT \geq \left(\int_T |v(S)|\,dT\right) \cos \delta.$$

Hence, $\mathfrak{X}(C_n) \leq \sec \delta$ and we may choose $M = \sec \delta$. Thus (7.1) is a possible sufficient condition on A.

It suffices that for some constant $b > 0$, $|V(C_n)| \geq b$ *for circles* C_n *in* A. For then

$$\mathfrak{X}(C_n) \leq b^{-1} \int_T |v(S)|\,dT \leq b^{-1} \pi^{n+1} d^{n(n+1)/2}$$

where d is the transfinite diameter of region α. Thus

$$M = \max\{1,\ b^{-1}\pi^{n+1} d^{n(n+1)/2}\}.$$

Let us next develop some sufficient conditions for regions A and B to have the second property required in Theorem I. These conditions are embodied in the following, cf. [8, pp. 52–57].

Theorem II. *Let* A *and* B *be bounded axisymmetric regions with* A *such that there exists a constant* $M \geq 1$ *with* $\mathfrak{X}(C_n) \leq M$ *for all choices of the* $n+1$ *circles* $C_n = \{(x_k, \rho_k)\} \subset \overline{A}$ *and for all* n. *Let*

$$D(x', \rho') = \{(x - x')^2 + (\rho - \rho')^2 \leq [a(x', \rho')]^2\}$$

be the smallest closed torus having as central circle (x', ρ') *and containing* B. *Let* Ω *be an axisymmetric region which contains the bounded closed set* $\Gamma = \{\bigcup D(x', \rho') : (x', \rho') \subset \overline{A}\}$. *Finally, let* $F(x, \rho)$ *be a function which is axisymmetric, harmonic in* Ω *and which is interpolated by the axisymmetric harmonic polynomial* $\Lambda_n(x, \rho; C_n)$ *on the* $n+1$ *circles* C_n. *Then* $\Lambda_n(x, \rho; C_n)$ *converges uniformly to* $F(x, \rho)$ *in* $\overline{B}$, *as* $n \to \infty$.

Proof. Let us denote by $\alpha, \beta, \gamma, \delta(x', \rho')$, and ω the meridian sections of the sets A, B, Γ, $D(x', \rho')$ and Ω respectively. Then $\delta_{kt} = \delta(x_k, \rho_k \cos t)$ is the smallest closed disk having its center at point $(x_k, \rho_k \cos t) \in \overline{\alpha}$ and containing $\overline{\beta}$. For all $t, \gamma \supset \bigcup_{k=0}^{n} \delta_{kt}$ and $\gamma \subset \omega$. Let us express $F(x, \rho)$ by (4.4) in terms of its associate $f(\sigma)$ which is holomorphic in ω. Formula (5.1) gives the nth degree polynomial $l_n(\sigma; S)$ which interpolates to $f(\sigma)$ at the points σ_k, $k = 0, 1, \cdots, n$.

In order to obtain the usual integral representation for the difference $[f(\sigma) - l_n(\sigma; S)]$, let us introduce (cf. [8, pp. 54–55]) the function $w = \phi(\sigma)$ which is holomorphic in the complement γ' of γ and which maps the simply connected region γ' conformally onto the disk $|w| > 1$ with $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$. Let κ_ϵ be the level curve $|\phi(\sigma)| = 1 + \epsilon$, where $\epsilon > 0$ and ϵ is chosen so small that $\kappa_\epsilon \subset \omega$. Let d_ϵ be the minimum distance from κ_ϵ to γ. Obviously $d_\epsilon > 0$.

By Hermite's formula (a corollary of the residue theorem) [2, p. 68]

$$f(\sigma) - l(\sigma, S) = \frac{1}{2\pi i} \int_{\kappa_\epsilon} \frac{f(s)\psi_n(\sigma)\, ds}{(s-\sigma)\psi_n(s)}, \quad \sigma \in \overline{\beta}, \tag{7.2}$$

where $\psi_n(\sigma) = \Pi_{k=0}^n (\sigma - \sigma_k)$, $\sigma_k \in \overline{\alpha}$. Let us define

$$\tau = \sup(|\sigma - \sigma_k|;\ \sigma \in \overline{\beta},\ \sigma_k \in \overline{\alpha}),$$
$$\mu = \sup(|f(s)|;\ s \in \kappa_\epsilon).$$

Since $\delta_{kt} \subset \gamma$, for all $\sigma \in \delta_{kt}$ and all $s \in \kappa_\epsilon$, $|\sigma - \sigma_k| + d_\epsilon \leq |s - \sigma_k|$. Since

$$(|\sigma - \sigma_k| + d_\epsilon)/|\sigma - \sigma_k| = 1 + d_\epsilon|\sigma - \sigma_k|^{-1} \geq 1 + (d_\epsilon/\tau),$$

$$|f(\sigma) - l_n(\sigma, S)| \leq \frac{\mu}{2\pi d_\epsilon} \left(\frac{\tau}{\tau + d_\epsilon}\right)^{n+1} \int_{\kappa_\epsilon} |ds| = a\nu^{n+1}, \tag{7.3}$$

where $a > 0$ and $0 \leq \nu < 1$. If we now substitute from (7.3) into (6.4), we find $|\Delta_n(x, \rho;\ C_n)| \leq aM\nu^{n+1}$. Clearly, given any $\epsilon > 0$, we can choose N so large that $|\Delta_n(x, \rho;\ C_n)| < \epsilon$ for all circles $C_n \subset \overline{A}$, $n \geq N$, and for any point on any circle $(x, \rho) \subset \overline{B}$.

Therefore, $\Lambda_n(x, \rho;\ C_n)$ converges to $F(x, \rho)$ uniformly on $\overline{B}$, as stated in Theorem II.

Theorem II may be restated with A and Ω given and B to be specified, or B and Ω given and A to be specified.

An immediate consequence of Theorem II is the following simpler result. Cf. [2, p. 81].

Corollary. *Let A and B be defined as in Theorem* II *and let Δ and Ω be bounded axisymmetric regions with $\overline{A} \subset B$, $\overline{B} \subset \Delta$, $\overline{\Delta} \subset \Omega$. Assume that* $0 < b < c$ *where*

$$b = \max\{[(x' - x)^2 + (\rho' - \rho)^2]^{1/2}: (x', \rho') \subset \partial A,\ (x, \rho) \subset \partial B\},$$
$$c = \min\{[(x' - x)^2 + (\rho' - \rho)^2]^{1/2}: (x', \rho') \subset \partial A,\ (x, \rho) \subset \partial \Delta\}.$$

Then, if $F(x, \rho)$ and $\Lambda_n(x, \rho;\ C_n)$ be specified as in Theorem II, $\Lambda_n(x, \rho;\ C_n)$ *converges uniformly to $F(x, \rho)$ in $\overline{B}$ as $n \to \infty$.*

Proof. The assumption $b < c$ ensures that $D(x', \rho') \subset \overline{\Delta}$ for every circle $(x', \rho') \subset \overline{A}$ and therefore that $\Gamma \subset \overline{\Delta}$. Thus, if Ω is chosen so that $\overline{\Delta} \subset \Omega$, then also $\Gamma \subset \Omega$ and Theorem II leads at once to the corollary.

Example. Let A be the ellipsoid of revolution and B the ball defined by the inequalities:

$$\overline{A} = \{(x^2 + \rho^2)^{1/2} + [(x - c)^2 + \rho^2]^{1/2} \leq b\}, \qquad b > c > 0,$$
$$\overline{B} = \{(x - c)^2 + \rho^2 \leq b^2\}.$$

Then Γ is the ball $x^2 + \rho^2 \leq (b + b)^2$ since $\{b + [(x - c)^2 + \rho^2]^{1/2}\}$ is the radius of the smallest circle having its center at any boundary point (x, ρ) of the elliptic meridian section of $\bar{A}$ and containing the meridian section of $\bar{B}$. The minor axis of ellipsoid A is $m = (b^2 - c^2)^{1/2}$, which can be taken small if necessary to satisfy the condition $\mathfrak{X}(C_n) \leq M$ for some $M \geq 1$. Finally, Ω can be taken as any ball $x^2 + \rho^2 < k^2$ with $k > (b + b)$.

8. **Another interpolation problem.** Instead of (1.1), let us impose upon $\Lambda_n(x_k, \rho_k; C_n)$ the conditions that

$$\Lambda_n^{(k)}(x_k, \rho_k; C_n) = F^{(k)}(x_k, \rho_k), \qquad k = 0, 1, 2, \cdots, n, \tag{8.1}$$

where supercript k signifies a kth partial derivative with respect to x. In eliminating the A_k from (2.2) subject to (8.1), we may use the identity obtained by differentiating (4.1) with respect to x:

$$P_n^{(1)}(x, \rho) = \frac{n}{\pi} \int_0^\pi (x + i\rho \cos t)^{n-1}\, dt = nP_{n-1}(x, \rho);$$

so that by induction

$$P_n^{(k)}(x, \rho) = n(n - 1) \cdots (n - k + 1)P_{n-k}(x, \rho). \tag{8.2}$$

If E denotes the determinant (2.4), we have now for $\Lambda_n(x, \rho; C_n)$ the equation:

$$\frac{\partial}{\partial x_1} \frac{\partial^2}{\partial x_2^2} \cdots \frac{\partial^n}{\partial x_n^n} E = 0. \tag{8.3}$$

The cofactor of Λ_n in the determinant on the left side of (8.3) is

$$W(C_n) = \frac{\partial}{\partial x_1} \frac{\partial^2}{\partial x_2^2} \cdots \frac{\partial^n}{\partial x_n^n} V(C_n) = \det \left\| \frac{\partial^j}{\partial x_j^j} P_k(x_j, \rho_j) \right\|$$
$$= \det \| k(k - 1) \cdots (k - j + 1)P_{k-j}(x_j, \rho_j) \| = 1!2! \cdots n!.$$

Defining

$$W_k(x, \rho; C_n) = \frac{\partial}{\partial x_1} \frac{\partial^2}{\partial x_2^2} \cdots \frac{\partial^{k-1}}{\partial x_{k-1}^{k-1}} \frac{\partial^{k+1}}{\partial x_{k+1}^{k+1}} \cdots \frac{\partial^n}{\partial x_n^n} V_k(x, \rho; C_n),$$

we obtain

$$\Lambda_n(x, \rho; C_n) = F(x_0, \rho_0) + [1!2! \cdots n!]^{-1} \sum_{k=1}^{n} F^{(k)}(x_k, \rho_k) W_k(x, \rho; C_n). \tag{8.4}$$

Similarly to (4.5), we may write

$$W_k(x, \rho; C_n) = \pi^{-n-1} \int_{T_k} w_k(S_k)\, dT_k \tag{8.5}$$

where

$$w_0(S_0) = \frac{\partial}{\partial\sigma_1}\frac{\partial^2}{\partial\sigma_2^2}\cdots\frac{\partial^n}{\partial\sigma_n^n} v(S_0),$$

(8.6)

$$w_k(S_k) = \frac{\partial}{\partial\sigma_1}\cdots\frac{\partial^{k-1}}{\partial\sigma_{k-1}^{k-1}}\frac{\partial^{k+1}}{\partial\sigma_{k+1}^{k+1}}\cdots\frac{\partial^n}{\partial\sigma_n^n} v(S_k), \qquad k = 1, 2, \cdots, n.$$

Thus we may rewrite (8.4) as

$$\Lambda_n(x, \rho; C_n) = F(x_0, \rho_0) + [1!2!\cdots n!\pi^{n+2}]^{-1} \int_{T^*} \sum_{k=1}^{n} f^{(k)}(\sigma_k) w_k(S_k)\, dT^*. \tag{8.7}$$

The quantities (8.6) also occur in the polynomial $l_n(\sigma; S) = \Sigma_{k=0}^n a_k\sigma^k$ which satisfies the conditions:

$$l_n^{(k)}(\sigma_k; S) = f^{(k)}(\sigma_k; S), \qquad k = 0, 1, \cdots, n. \tag{8.8}$$

On eliminating the a_k, we find that

$$l_n(\sigma, S) = f(\sigma_0) + [1!2!\cdots n!]^{-1} \sum_{k=1}^{n} f^{(k)}(\sigma_k) w_k(S_k). \tag{8.9}$$

This permits us to rewrite (8.7) as

$$\Lambda_n(x, \rho; C_n) = \pi^{-n-2} \int_{T^*} l_n(\sigma, S)\, dT^*. \tag{8.10}$$

A representation for $l_n(\sigma; S)$ alternative to (8.9) is

$$l_n(\sigma, S) = f(\sigma_0) + \sum_{k=1}^{n} f^{(k)}(\sigma_k) q_k(\sigma) \tag{8.11}$$

where

$$q_k(\sigma) = \int_{\sigma_0}^{\sigma} d\nu_1 \int_{\sigma_1}^{\nu_1} d\nu_2 \int_{\sigma_2}^{\nu_2} d\nu_3 \cdots \int_{\sigma_{k-1}}^{\nu_{k-1}} d\nu_k. \tag{8.12}$$

The kth degree polynomial $q_k(\sigma)$, is essentially the "Abel-Gontscharoff polynomial", defined by the equations [2, pp. 46–47]

$$q_k(\sigma_0) = q_k'(\sigma_1) = \cdots = q_k^{(k-1)}(\sigma_{k-1}) = 0, \qquad q_k^{(k)}(\sigma) \equiv 1. \tag{8.13}$$

Consequently, for $j = 0, 1, \cdots, n$

$$l_n^{(j)}(\sigma_j, S) = \sum_{k=j}^{n} f^{(k)}(\sigma_k) q_k^{(j)}(\sigma_j) = f^{(j)}(\sigma_j)$$

as required by (8.8).

Let us define

$$Q_k(x, \rho) = \pi^{-n-1} \int_{T_k} q_k(x + i\rho \cos t)\, dT_k. \tag{8.14}$$

From (8.10) and (8.12), we conclude that

$$\Lambda_n(x, \rho; C_n) = F(x_0, \rho_0) + \sum_{k=1}^{n} F^{(k)}(x_k, \rho_k) Q_k(x, \rho). \tag{8.15}$$

Regarding the convergence of $\Lambda_n(x, \rho; C_n)$ to $F(x, \rho)$, we may state the following theorem:

Theorem III. *Let A and B be bounded axisymmetric regions and the circles $C_n = \{(x_k, \rho_k)\} \subset \bar{A}$. Let Γ, Ω and $F(x, \rho)$ be defined as in Theorem* II. *Let $\Lambda_n(x, \rho; C_n)$ be the nth degree polynomial such that $\Lambda_n^{(k)}(x_k, \rho_k; C_n) = F^{(k)}(x_k, \rho_k)$, $k = 0, 1, \cdots, n$. Then $\Lambda_n(x, \rho; C_n)$ converges to $F(x, \rho)$ uniformly in $\bar{B}$, as $n \to \infty$.*

Proof. From (8.10) and (4.4), we observe that

$$F(x, \rho) - \Lambda_n(x, \rho; C_n) = \pi^{-n-2} \int_{T^*} [f(\sigma) - l_n(\sigma, S)]\, dT^*. \tag{8.16}$$

By the residue theorem

$$f(\sigma) - l_n(\sigma; S) = \frac{1}{2\pi i} \int_{\kappa_\epsilon} \frac{f(s)(\sigma - \sigma_0)(\sigma - \sigma_1)^2 \cdots (\sigma - \sigma_n)^n}{(s - \sigma)(s - \sigma_0)(s - \sigma_1)^2 \cdots (s - \sigma_n)^n}\, ds. \tag{8.17}$$

where κ_ϵ has the same meaning as for (7.2). Using the same notation as in (7.3), we infer that

$$|f(\sigma) - l_n(\sigma, S)| \le \frac{\mu}{2\pi d_\epsilon} \left(\frac{r}{r + d_\epsilon}\right)^N \int_{\kappa_\epsilon} |ds| = a\nu^N \tag{8.18}$$

where $a > 0$, $0 \le \nu < 1$ and $N = n(n+1)/2$. On using (8.18) in conjunction with (8.16), we infer that $|F(x, \rho) - \Lambda_n(x, \rho; C_n)| \le a\nu^N$ and thus complete the proof of Theorem III. [A corollary similar to that for Theorem II is also valid.]

9. Expansion about a single circle. Let us finally seek the axisymmetric harmonic polynomial $\Lambda_n(x, \rho; x_0, \rho_0)$ which in a single circle (x_0, ρ_0) has $(n+1)$-fold coincidence with an axisymmetric harmonic function $F(x, \rho)$ in the sense that

$$(9.1)\qquad \Lambda_n^{(j)}(x_0, \rho_0; x_0, \rho_0) = F^{(j)}(x_0, \rho_0), \qquad j = 0, 1, \cdots, n.$$

This corresponds to (8.1) with all $n+1$ circles C_n coalescing in a single circle (x_0, ρ_0). Consequently, from (8.4)

$$(9.2)\quad \Lambda_n(x, \rho; x_0, \rho_0) = F(x_0, \rho_0) + [1!2! \cdots n!]^{-1} \sum_{j=1}^{n} F^{(j)}(x_0, \rho_0) U_j(x, \rho; x_0, \rho_0)$$

where $U_k(x, \rho; x_0, \rho_0) = \{W_k(x, \rho; C_n): (x_j, \rho_j) = (x_0, \rho_0), \ j = 1, 2, \cdots, n\}$. On developing this determinant in the form

$$U_j(x, \rho; x_0, \rho_0) = \sum_{k=0}^{n} c_{jk} P_k(x, \rho),$$

one finds that $c_{jk} = 0$ for $k > j$. That is, $U_j(x, \rho; x_0, \rho_0)$ is a polynomial of degree j. By differentiating the determinant ν times with respect to x, we verify immediately that

$$(9.3)\qquad U_j^{(\nu)}(x_0, \rho_0; x_0, \rho_0) = 0, \qquad \nu = 0, 1, 2, \cdots, j-1,$$

$$(9.4)\qquad U_j^{(j)}(x_0, \rho_0; x_0, \rho_0) = 1!2! \cdots n!.$$

Thus we verify at once that $\Lambda_n(x, \rho; x_0, \rho_0)$ as given by (9.2) does satisfy all the conditions (9.1).

Furthermore, since each $P_k(x, \rho)$ is an even function of ρ, we infer that $U_j^{(\nu)}(x_0, \rho; x_0, \rho_0)$ is divisible by $(\rho^2 - \rho_0^2)$ for $\nu = 0, 1, \cdots, j$. Taken with (9.3) and (9.4), this information means that we can write $U_j(x, \rho; x_0, \rho_0)$ in the form

$$(9.5)\qquad \begin{aligned} &U_j(x, \rho; x_0, \rho_0) \\ &= [1!2! \cdots n!/j!](x - x_0)^j + \sum_{\nu=1}^{[j/2]} d_{j\nu}(x - x_0)^{j-2\nu}(\rho^2 - \rho_0^2)^{\nu}. \end{aligned}$$

A more direct approach to an expansion about the circle (x_0, ρ_0) is, using (4.4) and assuming $f(\sigma)$ to be holomorphic in the closure of ω, to write

$$(9.6)\qquad F(x_0 + b, \rho_0 + k) = \frac{1}{\pi}\int_0^{\pi} f[(x_0 + i\rho_0 \cos t) + (b + ik \cos t)]\, dt$$

with $c^2 = b^2 + k^2 < a_0^2$ where a_0 is the shortest distance of points $\sigma_0 = x_0 + i\rho_0 \cos t$, $0 \le t \le \pi$, to the boundary $\partial\omega$ of ω. Setting $\tau = b + ik \cos t$, we have

$$(9.7)\qquad f(\sigma_0 + \tau) = f(\sigma_0) + f'(\sigma_0)\tau + \cdots + f^{(n)}(\sigma_0)\tau^n/n! + r_n(\sigma_0, \tau)\tau^{n+1}$$

where $r_n(\sigma_0, \tau) = (1/2\pi i)\int_{K_\epsilon} f(s)(s - \sigma_0)^{-n-2}(s - \sigma_0 - \tau)^{-1} ds$ and K_ϵ is a circle $|s - \sigma_0| = b_0$ with $c < b_0 < a_0$.

Let us define the product $H_1(x_1, \rho_1) * H_2(x_2, \rho_2)$ of two axisymmetric harmonic functions, for which the associates are $h_1(\sigma_1)$ and $h_2(\sigma_2)$ respectively, as

$$(9.8)\quad H_1(x_1, \rho_1) * H_2(x_2, \rho_2) = \frac{1}{\pi}\int_0^\pi h_1(x_1 + i\rho_1 \cos t) h_2(x_2 + i\rho_2 \cos t)\, dt.$$

This product is an axisymmetric harmonic function if $x_1 = x_2$ and $\rho_1 = \rho_2$, but in general it is not a harmonic function.

Using (9.6), (9.7) and (9.8), we may now write

$$(9.9)\quad F(x_0 + h, \rho_0 + k) = F_n(x_0 + h, \rho_0 + k) + R_n(x, \rho; x_0, \rho_0) * P_{n+1}(h, k)$$

where

$$(9.10)\qquad F_n(x_0 + h, \rho_0 + k) = F(x_0, \rho_0) + \sum_{j=0}^{n} (j!)^{-1} F^{(j)}(x_0, \rho_0) * P_j(h, k)$$

and

$$(9.11)\quad \begin{aligned} R_n(x_0 + h, \rho_0 + k; x_0, \rho_0) = \int_0^\pi \int_{K_\epsilon} & f(s)(s - x_0 - i\rho_0 \cos t)^{-n-2} \\ & \cdot [s - (x_0 + h) - i(\rho_0 + k) \cos t]^{-1}\, ds\, dt. \end{aligned}$$

Since $P_j^{(\nu)}(0, 0) = 0$ or $j!$ according as $\nu < j$ or $\nu = j$, it follows that $F_n^{(\nu)}(x_0 + 0, \rho_0 + 0) = F^{(\nu)}(x_0, \rho_0)$ and thus $F_n(x_0 + h, \rho_0 + k) = \Lambda_n(x_0 + h, \rho_0 + k; x_0, \rho_0)$.

Let us next evaluate $|R_n(x_0 + h, \rho_0 + k; x_0, \rho_0)|$, setting $\mu = \{\max |f(\sigma)| : \sigma \in \partial\omega\}$. Thus

$$|R_n(x_0 + h, \rho_0 + k; x_0, \rho_0)| \le \mu b_0^{-n-1}(b_0 - c)^{-1}.$$

Also

$$|P_{n+1}(h, k)| \le c^{n+1} |P_{n+1}(h/c)| \le c^{n+1}.$$

In view of these results and (9.9), we may now state the following:

Theorem IV. *Let Ω be an axisymmetric region in whose closure $F(x, \rho)$ is an axisymmetric harmonic function. Let (x_0, ρ_0) be any circle in Ω. Then the nth degree axisymmetric harmonic polynomial $\Lambda_n(x, \rho; x_0, \rho_0) \equiv F_n(x_0 + h, \rho_0 + k)$ given by (9.2) or (9.10) has the property:*

$$\Lambda_n^{(j)}(x_0, \rho_0; x_0, \rho_0) = F^{(j)}(x_0, \rho_0), \qquad j = 0, 1, \cdots, n,$$

where the superscript (j), denotes the jth derivative with respect to x. Furthermore, $\Lambda_n(x, \rho; x_0, \rho_0)$ converges uniformly to $F(x, \rho)$ in any torus $(x - x_0)^2 + (\rho - \rho_0)^2 \le c^2$ that is contained in Ω, as $n \longrightarrow \infty$.

The nth degree polynomial $F_n(x_0 + h, \rho_0 + k)$ in h and k is clearly the analogue of the Taylor polynomial for a holomorphic function $f(\sigma)$, $\sigma \in \mathbf{C}$. It is however not harmonic in h and k in general, but it is axisymmetric harmonic in $x = x_0 + h$ and $\rho = \rho_0 + k$.

10. **Generalization to $\mathbf{R}^N$.** We now propose to generalize the preceding results to N dimensions. Let us consider functions F of N real variables $\xi_1, \xi_2, \cdots, \xi_N$ that depend only upon the two quantities x and ρ where

$$x = \xi_1, \quad \rho^2 = \xi_2^2 + \xi_3^2 + \cdots + \xi_N^2, \quad N \geq 3. \tag{10.1}$$

The corresponding Laplace's differential equation $\nabla^2 F = 0$ is then (cf. [5, p. 167])

$$(\partial/\partial x)(\rho^{N-2}\partial F/\partial x) + (\partial/\partial\rho)(\rho^{N-2}\partial F/\partial\rho) = 0 \tag{10.2}$$

and its solutions are the axisymmetric harmonic functions in $\mathbf{R}^N$. By a circle (x_k, ρ_k) we mean the locus of points in $\mathbf{R}^N$ that satisfy simultaneously the two equations $x = x_k, \rho = \rho_k$ (cf. [5, pp. 151–152]), and by an axisymmetric region $\Omega \subset \mathbf{R}^N$ we mean one such that, if circle $(x_0, \rho_0) \subset \Omega$, then also circle $(x_0, \rho) \subset \Omega$ for $0 \leq \rho \leq \rho_0$. We propose the same problems in $\mathbf{R}^N$ as were stated in §1 for $\mathbf{R}^3$.

It is well known that every axisymmetric harmonic polynomial in $\mathbf{R}^N$ may be written as a linear combination of the polynomials $P_k(x, \rho)$ defined by the equation

$$P_k(x, \rho) = r^k P_k^{\mu}(x/r), \qquad \mu = (N-2)/2, \tag{10.3}$$

where $r^2 = x^2 + \rho^2$.

The function $P_k^{(\mu)}(u)$ is the so-called Gegenbauer or ultraspherical harmonic polynomial given by [7, pp. 81–85]

$$P_k^{(\mu)}(u) = \sum_{j=0}^{[k/2]} (-1)^j \gamma_{kj} u^{k-2j} \tag{10.4}$$

with coefficients expressed in terms of the gamma function as $\gamma_{kj} = 2^{k-2j}\Gamma(k-j+\mu)/\Gamma(\mu)\Gamma(j+1)\Gamma(k-2j+1)$. Corresponding to equation (4.1), we now have (cf. [7, p. 97])

$$P_k(x, \rho) = p_k \int_0^{\pi} (x + i\rho\cos t)^k \sin^{N-3} t\, dt \tag{10.5}$$

where

$$p_k = 2^{3-N}\Gamma(\mu)^{-2}[(k+N-3)!/k!]. \tag{10.6}$$

We note that $P_k^{(\mu)}(1) = P_k(1, 0) = (k + N - 3)!/[k!(N-3)!]$. When $N = 3$ and thus $\mu = \frac{1}{2}$, $P_k^{(\mu)}(u)$ clearly reduces to the Legendre polynomial of degree k. From (2.2), (10.3) and (10.5) we derive the relation

$$\Lambda_n(x, \rho; C_n) = \int_0^\pi \lambda_n(x + i\rho \cos t) \sin^{N-3} t \, dt \tag{10.7}$$

where the polynomial $\lambda_n(\zeta) = \Sigma_{k=0}^n p_k A_k \zeta^k$, $\zeta \in \mathbf{C}$, is called the associate of $\Lambda_n(x, \rho; C_n)$. This suggests the representation

$$F(x, \rho) = \int_0^\pi f(x + i\rho \cos t) \sin^{N-3} t \, dt \tag{10.8}$$

where $f(\zeta)$ is a holomorphic function of the complex variable $\zeta = \xi + i\eta$ in the meridian section ω of Ω; that is, on the set of the intersection points of Ω with a plane

$$\xi_j = \rho k_j, \quad j = 2, 3, \cdots, N, \quad k_2^2 + k_3^2 + \cdots + k_N^2 = 1.$$

Indeed, if the origin is contained in Ω, the series $F(x, \rho) = \Sigma_{k=0}^\infty c_k P_k(x, \rho)$ is valid uniformly within N dimensional ball $x^2 + \rho^2 \le r_0^2$ contained in Ω. Thus the series $f(\zeta) = \Sigma_{k=0}^\infty p_k c_k \zeta^k$ is valid uniformly within the disk $|\zeta| \le r_0$ contained in ω, since the p_k given by (10.6) are uniformly bounded for all k. The extension of the representation (10.8) for $F(x, \rho)$ in all of Ω and for $f(\zeta)$ in all of ω is by harmonic and analytic continuations respectively.

Furthermore, from (10.8) it follows that

$$f(\zeta) = F(\zeta, 0) \Big/ \int_0^\pi \sin^{N-3} t \, dt. \tag{10.9}$$

That is, we may regard $f(\zeta)$ as the analytic continuation of the function given by the right side of (10.9), from the real ζ to complex ζ.

The determination of the A_k in (2.2) now proceeds as in §2, with the result given by equation (2.7). In order to generalize equation (4.6), we add the following to the notation given in §4:

$$M(T) = \prod_{k=0}^n p_k \sin^{N-3} t_k, \quad M(T_k) = [M(T)]_{t_k = t}, \quad M(T^*) = M(T) \sin^{N-3} t.$$

Then since $V(C_n) = \det \|P_k(x, \rho)\| = \int_T v(S) M(T) dT$, we find

$$\Lambda_n(x, \rho; C_n) = \frac{\Sigma_{k=0}^n \int_{T^*} f(\sigma_k) v(S_k) M(T^*) \, dT^*}{\int_T v(S) M(T) \, dT}.$$

Finally, if we introduce the Lagrange interpolation polynomial $l_n(\sigma; S)$ given by equation (5.1), we obtain the expression

$$\Lambda_n(x, \rho; C_n) = \frac{\int_{T^*} l_n(\sigma; S)\nu(S)M(T^*)\,dT^*}{\int_T \nu(S)M(T)\,dT}.$$

The details regarding the convergence of $\Lambda_n(x, \rho; C_n)$ to $F(x, \rho)$ carry over to $\mathbf{R}^N$ essentially as in §6. In fact, if we now define, instead of (6.3),

$$\mathfrak{X}(C_n) = \int_T |\nu(S)|M(T)\,dT \Big/ \left|\int_T \nu(S)M(T)\,dT\right|,$$

the theorems given in §6 and §7 remain valid for axisymmetric harmonic functions in $\mathbf{R}^N$. Similarly, we may modify the material in §8 and §9 so that it also remains valid for $\mathbf{R}^N$.

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